A Model of Interface Growth
with non-Burgers Dynamical Exponent

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Abstract

We define a new model of interface roughening which has the property that the minimum of interface height is conserved locally during the growth. This model corresponds to the limit $q \to \infty$ of the q-color dimer deposition-evaporation model introduced by us earlier [Hari Menon M K and Dhar D 1995 *J. Phys. A: Math. Gen.* **28** 6517]. We present numerical evidence from Monte Carlo simulations and the exact diagonalization of the evolution operator on finite rings that this model is not in the universality class of the Kardar-Parisi-Zhang interface growth model. The dynamical exponent z in one dimension is larger than 2, with $z \approx 2.5$. And there are logarithmic corrections to the scaling of the gap with system size. Higher dimensional generalization of the model is briefly discussed.

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1   Introduction

The structure of growing interfaces has been a major subject of study in recent years [1, 2]. Most of these studies concern the large scale properties of a moving interface between two phases. The recent spurt of interest in this subject started with the proposal of a nonlinear evolution equation for the interface height $h(\vec{x}, t)$ by Kardar, Parisi and Zhang (KPZ) one decade ago [3]. KPZ argued that consistent with the symmetries of the interface, the lowest order nonlinear term in the growth equation is proportional to $(\nabla h)^2$. Hence in a large number of physical situations the growing interface can be described by a noisy Burgers equation. In $1+1$ dimensions this nonlinear term is the most relevant perturbation, and it gives rise to a dynamical universality class characterized by the value of the dynamical exponent $z = 3/2$. Addition of higher order nonlinear terms, consistent with the symmetries, will not change the value of z.

Though this is indeed the case found in many physical situations, there are cases where the correlations of the interface are not described by KPZ exponents [4, 5]. For example, if there are some constraints present for the growth, we can expect the relaxation of the interface to become slower.

Some generalizations of the KPZ equation where the scalar height variable is replaced by an N-component vector [6], or an $N \times N$ matrix [7] have been studied. But in both these cases it was found that the value of z remains unchanged in $1+1$ dimensions.

In this context, it is interesting to note that recently a class of deposition-evaporation models has been studied which in $1+1$ dimensions can be mapped to interface roughening models where the height variables are 2×2 matrices [8-14]. Monte Carlo simulations and numerical diagonalization studies have shown that these models do not belong to the KPZ dynamical universality class [12, 13]. The numerical studies strongly suggests that $z \approx 5/2$. An example of this is the trimer deposition-evaporation (TDE) model [8]. Numerical diagonalization studies on small lattices shows that $z \approx 2.5$ for this model [12]. Another example is the q-color dimer deposition-evaporation (qDDE) model [13]. For the case of $q = 2$, qDDE model reduces to the Heisenberg model which is exactly solvable [8], and the corresponding value of z is 2. For $q > 2$, numerical studies suggests that this model is in the same universality class as the TDE model.

Though there is fairly strong numerical evidence for this new universality class with $z \approx 2.5$, so far there have been no models where the precise value of the dynamical exponent can be established more convincingly. Obukhov et al have given scaling arguments that $z = 5/2$ in the context of a related model for diffusion of ring polymers in gels [15]. Using similar, but somewhat more careful arguments, Alon and Mukamel reached the same conclusion [16]. It is thus of some interest to look for simpler models in this universality class, for which analytical results can be obtained. It is well known that many spin models become simpler, and in some cases even exactly solvable, in the limit when the number of components of the spin tends to infinity [17]. This has motivated us to the study of the qDDE model in the limit of $q \to \infty$. Our main result is that in this limit, the qDDE model reduces to a simpler interface growth model with scalar heights. *This interface model has an interesting constraint that the minimum height of the interface is conserved locally during the growth.* We have studied this simpler model numerically, but have not been able to solve it analytically so far.

This paper is organized as follows: In section 2 we define the interface growth model, and write down the stochastic matrix as the Hamiltonian of a quantum mechanical spin chain. We argue that conservation of minimum height locally in the model makes the relaxation slow. In section 3 we calculate the average height of the interface in the steady state for a ring of size L, and show that this grows as $L^{1/2}$ for large L. In section 4 we recapitulate the definition of the qDDE model and briefly list its known properties. In section 5 we study the dynamics of the qDDE model in the limit of large q. We show that in this limit the model simplifies, and for time scales much greater than $1/q$ the model can be described by an effective Markovian dynamics. In section 6 we establish an equivalence between this effective dynamics and the dynamics of the interface model defined in section 2. In section 7 we study this interface model using both Monte Carlo simulations and numerical diagonalization of the stochastic matrix for small lattice sizes. These studies show that this interface model is in the same universality class as the TDE model and the qDDE model with finite $q > 2$. In section 8 we propose some higher dimensional generalizations of this interface model, which still conserve the minimum height of the interface locally.

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2   Definition of the Interface model

We consider the interface model on a one-dimensional lattice of size L. At any given time, the interface is specified by the integer height h_i at each site i of the lattice. The heights are assumed to obey the restricted solid-on-solid (RSOS) condition

$$h_{i+1} - h_i = \pm 1 \tag{1}$$

for all i and at all times. The interface evolves by the following Markovian local dynamics: at every site the interface height can be changed from h_i to either $h_i + 2$ or $h_i - 2$ with some rates, provided this will not violate the RSOS condition. *The rates for the transition $h_i \rightarrow h_i \pm 2$ depends on the next neighbouring heights h_{i-2} and h_{i+2}.*

Equivalently, we can specify the interface in terms of slope variables $n_i = [h_{i+1} - h_i + 1]/2$, which takes only values 0 and 1. Then change in the height at site i corresponds to the exchange of the variables n at site i and $i - 1$. If we think of n's as occupation variables of a hard core lattice gas, this corresponds to the well-known exclusion process, with hopping rates between sites i and $i + 1$ depends on the occupation at sites $i - 1$ and $i + 2$. In our model we assume that the rate for the rightward and leftward hoppings (or $h_i \rightarrow h_i + 2$ and $h_i \rightarrow h_i - 2$) are the same. Then there are 4 hopping rates, depending on the four possible states of the sites $i - 1$ and $i + 2$. Let us call these rates $\lambda_1, \lambda_2, \lambda_3$ and λ_4. These are shown in figure 1.

When the hopping rate between sites i and $i + 1$ is independent of n_{i-1} and n_{i+2} (simple exclusion process), the model is exactly solvable. The corresponding interface growth is described by the Hammersley-Edwards-Wilkinson equation [21, 22]. For this model the dynamical exponent $z = 2$. Further if the forward and backward transition rates are not equal (asymmetric exclusion process), the model would corresponds to a growing interface that is described by the KPZ equation, for which $z = 3/2$, [2, 3]. The asymmetric version of our model is interesting, but has not been studied in detail.

At time $t = 0$, the interface height h_i is given to be 0 if i is odd, and 1 if i is even. At the boundaries, we can work with fixed boundary conditions corresponding to choosing the height to be 0 at $i = 0$ and $L + 1$ at all times. Alternatively, we shall use periodic boundary conditions so that $h_{L+1} = h_1$.
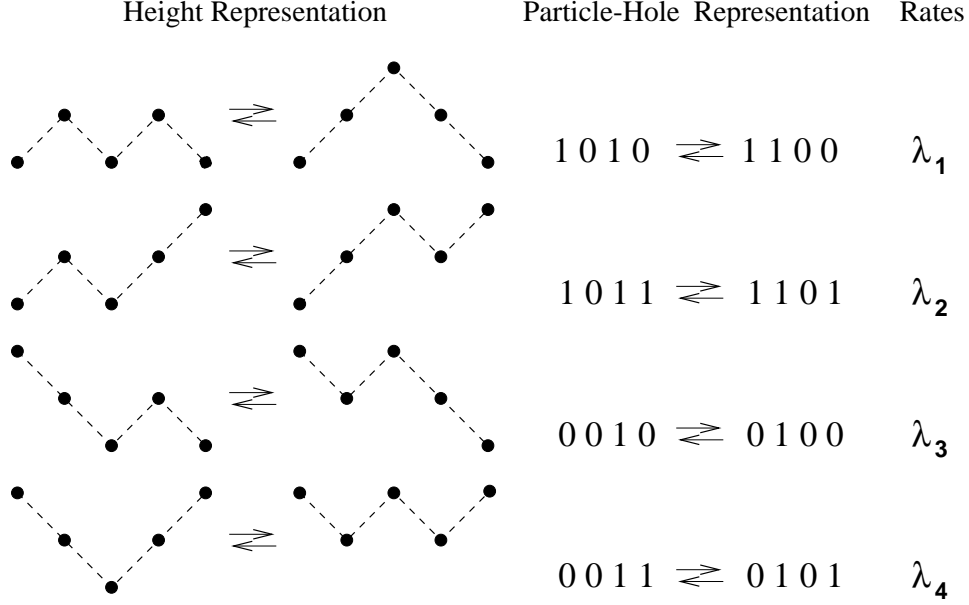
Height Representation	Particle-Hole Representation	Rates

$1\,0\,1\,0\ \rightleftarrows\ 1\,1\,0\,0 \qquad \lambda_1$

$1\,0\,1\,1\ \rightleftarrows\ 1\,1\,0\,1 \qquad \lambda_2$

$0\,0\,1\,0\ \rightleftarrows\ 0\,1\,0\,0 \qquad \lambda_3$

$0\,0\,1\,1\ \rightleftarrows\ 0\,1\,0\,1 \qquad \lambda_4$

Figure 1: Transition rates of the interface growth model.

This choice of rates satisfies the detailed balance condition, and in the steady state, all allowed configurations occur with equal weight. The case when forward and backward jumps occur with unequal rates is of interest, but will not be discussed here. If $\lambda_i = 1$ for all i, then this model corresponds to the Rouse model of polymer dynamics [18]. If all the rates λ_i, $i = 1$ to 4 are nonzero, then it is easy to see that qualitative behavior of the relaxation is not changed much. Let $\Delta E(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is the smallest nonzero eigenvalue of the relaxation matrix. The relaxation matrix is symmetrical, and can be thought of as the force matrix of a system of mass points connected by springs with spring-constants λ_i. As the eigenfrequencies are nondecreasing functions of the spring constants, it follows that

$$\frac{d}{d\lambda_i}\Delta E \geq 0 \qquad (2)$$

It is then easy to deduce that

$$\lambda_{min}\Delta E(1,1,1,1) \leq \Delta E(\lambda_1, \lambda_2, \lambda_3, \lambda_4) \leq \lambda_{max}\Delta E(1,1,1,1) \qquad (3)$$

where λ_{min} and λ_{max} are the smallest and largest values in a given $\{\lambda_i\}$.

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Now, $\Delta E(1, 1, 1, 1)$ is the lowest relaxation rate in the Rouse model, and it is known that for a ring of size L, it decreases as L^{-2} for L large. This implies that so long as λ_{min} and λ_{max} are finite, $\Delta E(\{\lambda_i\})$ decreases as L^{-2} for large L, and the dynamical exponent z remains 2. However, if λ_{min} becomes 0, then there is a possibility that we get a different universality class. *In this paper, we study the case when $\lambda_4 = 0$, and all other λ's are nonzero.* [The case $\lambda_1 = 0$, all other λ's nonzero is equivalent to this]. To keep the interface model transition rules left-right symmetric, we assume in addition that $\lambda_2 = \lambda_3$.

An important consequence of $\lambda_4 = 0$ is that minimum of $\{h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}\}$ is conserved during a change of heights at i. This implies, in particular, that this interface dynamics conserves minimum of the full interface profile $\{h_i\}$. *The constraint that minimum height is locally conserved is a strong constraint that makes the dynamics slower than the Rouse dynamics.* Note that the conclusion $z \geq 2$ for our model follows immediately from the inequality 3.

As an illustration of this, consider the transition in figure 2 from the initial configuration I to final configuration F. Both these configurations are allowed, but to go from I to F, it takes a long process of restructuring. The shortest route is to first completely erase one of the two 'mountains', and then rebuild the final structure F from that. It is easy to convince oneself that other routes, essentially equivalent and requring as many steps, are possible, but none requiring fewer can be found. For a pair of mountains of size L each, to go from I to F it takes order L^2 steps. In contrast in Rouse dynamics, F can be reached from I in one step.

The constraint that minimum of height is locally conserved is approximately realized in some physical situations, which give rise to surface morphology sometimes known as 'wedding cake' morphology with a surface roughness exponent α close to 1 [19]. These are understood to be due to the existence of step-edge energy barriers of the Ehrlich-Schwoebel type, which inhibit jumps of atoms from higher steps to lower steps [20]. Clearly, this makes growth at height minima less likely. In the Monte Carlo simulations of Krug, the surfaces generated have deep ridges which seem to survive for a long time. Our model is simpler than his, but generates shapes of surfaces qualitatively similar. In particular, in both cases fluctuations in height are of the same order as mean height.

It is quite striaghtforward, but instructive, to write the stochastic matrix of the interface model as a quantum Hamiltonian $\hat{H}$. We consider a chain of L quantum mechanical spins

<div align="center">I   F</div>
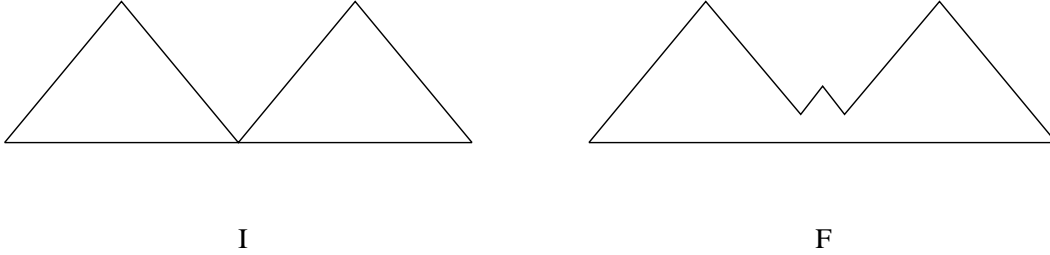
Figure 2: An example of two configurations where it takes order of L^2 steps to reach one from the other.

$\{S_i\}, i = 1$ to L. To each configuration $\{n_i\}$, we associate a spin configuration $\{S_i^z\}$ by the rule $n_i = (1 + S_i^z)/2$. Then it is easily seen that [23] the quantum-mechanical Hamiltonian corresponding to our model is

$$\hat{H} = \sum_{i=1}^{L} f[\hat{S}_{i-1}^z, \hat{S}_{i+2}^z] \, (\hat{\vec{S}}_i \hat{\vec{S}}_{i+1} - 1), \tag{4}$$

where the function f takes arbitrary positive values except that $f(-1, +1) = 0$. This is a four spin interaction Hamiltonian, which is not yet tractable analytically. Note that the Hamiltonian is not left-right symmetric though the original height model is. This is due to the fact the transformation from h_i to n_i breaks the reflection symmetry. It is invariant under the simultaneous interchange $x \longleftrightarrow -x$ and $n \longleftrightarrow 1 - n$.

3 Steady State Properties of the Interface

Starting from an initial profile, the interface will grow until the system reaches a steady state. The average interface height in the steady state (the saturation height) can be computed exactly by mapping the interface profiles to paths of a random walk that is not allowed to cross the origin. Since the interface heights satisfy $h_{i+1} = h_i \pm 1$, the set $\{h_i\}$ forms the path of a random walk if we imagine h and i as space and time co-ordinates. The constraint that the walk is not allowed to cross the origin comes from the fact that the interface height at any point can not be negative.

<div align="center">7</div>

Since the rules of the interface dynamics obey detailed balance, all accesible interface configurations have equal weight in the steady state. The average height of the interface then corresponds to the average displacement of the random walk from the origin, this can be computed exactly as follows: let us assume periodic boundary conditions and $L = 2n$, an even integer. We pick a point at random on the ring, and call it the origin. The number of configurations where height at the origin is h is given by the number of paths from $(i = 0, h)$ to $(i = 2n, h)$ which touches the line $h = 0$ but donot fall below. Let us denote this number by $\tilde{N}[(0, h); (2n, h)]$. If we denote $N_j[(0, h); (2n, h)]$ the number of paths from $(0, h)$ to $(2n, h)$ which donot fall below the line $h = j$,

$$\tilde{N}[(0, h); (2n, h)] = N_0[(0, h); (2n, h)] - N_1[(0, h); (2n, h)] \tag{5}$$

From the well known reflection principle [24],

$$N_j[(0, h); (2n, h)] = M[(0, h); (2n, 2j - h - 2)] \tag{6}$$

where $M[(0, h); (2n, h')]$ denotes the number of paths from $(0, h)$ to $(2n, h')$ with no constraint on the path. Hence

$$\tilde{N}[(0, h); (2n, h)] = M[(0, h); (2n, -h)] - M[(0, h); (2n, -h - 2)] \tag{7}$$

Since

$$
\begin{aligned}
M[(0, h); (2n, h')] &= \binom{2n}{\frac{|h' - h|}{2} + n}, \\
\tilde{N}[(0, h); (2n, h)] &= f(h) - f(h + 1),
\end{aligned}
\tag{8}
$$

where,

$$f(h) = \binom{2n}{n + h} \tag{9}$$

Therefore the average height is given by

$$
\begin{aligned}
\langle h \rangle &= \frac{\sum_{h=0}^{n} h \ [f(h) - f(h + 1)]}{\sum_{h=0}^{n}[f(h) - f(h + 1)]} \tag{10} \\
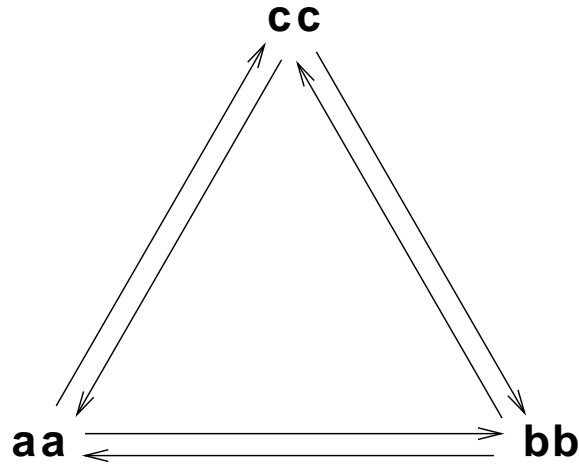&= \frac{1}{2}[2^{2n} / \binom{2n}{n} - 1] \tag{11}
\end{aligned}
$$

Figure 3: Allowed transitions in the qDDE model for $q = 3$

which increases as $\sqrt{\frac{\pi}{8}}\ L^{1/2}$ for large L. For an infinite system this means that starting from a profile which has the minimum possible heights ($\{h_i\} = \{0, 1, 0, 1, \ldots\}$), the average interface height will always grow. However due to the conservation of minimum height, it will always be tethered to $h = 0$ at least at one point. This is an interesting example of a growing interface with completely reversible rules.

4   A Brief Review of the qDDE Model

The qDDE model is defined as follows: Consider a d-dimensional lattice, where at each site there is a discrete variable which can take q distinct discrete values (colors). The system undergoes a continuous time Markovian dynamics with the update rule that with rate 1 two neighboring spins having the same color, can simultaneously change their color to any of the other $(q - 1)$ colors. For example consider the case of $q = 3$, and let a, b and c denote the 3 colors. An aa pair can become a bb pair or a cc pair with rate 1. In the same manner bb and cc pairs can change their color. The dynamical rule for this case can be stated as in Figure 3.

The qDDE model has been studied in detail in $1 + 1$ dimensions [13]. The main feature of the model is that its phase space breaks up into an exponentially large number of dynamically

disconnected sectors. The number of sectors scales as $(q-1)^L$ with system size. This strong nonergodic behavior is due to the presence of a conserved quantity in the model, called *irreducible string* (IS) which is defined as follows: A configuration of the qDDE model on a linear chain of length L can be represented by a string of L characters where the ith character represents the color of the ith spin. From this string delete all pairs of adjacent characters that are the same. Repeat this procedure on the resulting string until a string with no immediate repetition of characters is obtained. This string, whose length can not be reduced further by this reduction algorithm is the IS corresponding to given configuration. It can be shown that IS is conserved during the qDDE dynamics and each sector has a distinct IS [13]. As a consequence, IS can be used to uniquely specify a sector. The special sector for which the irreducible string is a null string (the original string is completely reducible) is called the *null sector* and is found to have a slower relaxation to the steady state.

In any given sector, in the steady state all the configurations of that sector occur with equal probability. This follows from detailed balance. The number of configurations in each sector can be computed exactly, and it typically grows as $\exp(L)$. For example, in the null sector the number of configurations grows as $[4(q-1)]^{L/2}$.

In addition to IS, the qDDE model in one-dimension has another set of conserved quantities which corresponds to a symmetry called *recoloring symmetry* in the model. A qDDE configuration can be represented by the configuration of a polymer chain on a q-coordinated Bethe lattice. To see this, consider a q-coordinated Bethe lattice with its bonds colored with q different colors such that all the bonds meeting at any given site have different colors. We define the polymer chain corresponding to a given qDDE configuration as the L step walk which starts from the origin and follows the sequence of bonds such that their color is in the same sequence as the colors in the qDDE configuration from site 1 to site L. Thus there is a one-to-one correspondence between the L step polymer chain configurations on the Bethe lattice and the configurations of the qDDE model. In this representation, the qDDE dynamics corresponds to a kind of reptation motion of the polymer chain on the Bethe lattice. A kink, consisting of two adjacent steps of the polymer chain where an immediate retraversal occurs, can jump to one of the neighbouring sites on the Bethe lattice. The rules of the dynamics of the polymer chain is independent of the color of the bonds and hence recoloring of the bonds is a symmetry of the model. Note that this recoloring symmetry is not just the symmetry

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under permutations of colors which corresponds to a global rotation in the color space. It allows a local recoloring at each bond, and thus is more like a gauge symmetry [1]. Using the recoloring symmetry a large number of eigenstates of the stochastic matrix of the model can be computed exactly [13]. However these eigenstates are not the low lying eigenstates, which determine the long time behavior and hence the dynamical exponent z.

The decay of spin-spin correlation functions in the steady state is found to be sector dependent for the qDDE model. The equal time two point correlation function can be determined analytically, and in the null sector it decays as $r^{-3/2}$. Hence we expect the time dependent spin-spin auto-correlation function to decay as $t^{-3/2z}$ asymptotically in this sector. Numerical diagonalization studies of the stochastic matrix shows that the dynamical exponent $z \approx 2.5$ in the null sector. In other sectors the decay of the autocorrelation function can be determined using the relationship to hard core random walk with conserved spin (HCRWCS) model introduced in [11]. HCRWCS model corresponds to a simple exclusion process with the particles carrying a spin label on each of them. Hence for these non-null sectors the dynamical exponent is 2.

5   Dynamics in the Large q Limit

Consider the qDDE model on a chain of length L with open boundary conditions. Here we restrict to only the null sector but our following analysis is equally applicable to other sectors. In terms of polymer chain configurations, we consider the case where both the end points of the chain are fixed to be at the origin. We divide the set of configurations in this sector into equivalence classes such that all configurations in a given equivalence class are related to each other by recoloring symmetry. As an example, in table 1 all the equivalence classes are shown for $L = 6$ along with the number of configurations in each class. Each class has a unique topology of polymer chain configuration on the Bethe lattice. As an example the polymer chain configurations of equivalence classes $\mathbb{C}_1$, $\mathbb{C}_9$ and $\mathbb{C}_{10}$ are shown in figure 4.

Note that for $\mathbb{C}_8 - \mathbb{C}_{12}$, each bond of the Bethe lattice is traversed either twice or not at all by the polymer chain and these classes have more number of configurations than the other

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[1]Of course, we cannot change the color only at a finite number of sites of the Bethe lattice without violating the constraint that all bonds meeting at a site have different colors

Equivalence Class	Representative Element	Number of Configurations
$\mathbb{C}_1$	aaaaaa	q
$\mathbb{C}_2$	aaaabb	q (q-1)
$\mathbb{C}_3$	aaabba	q (q-1)
$\mathbb{C}_4$	aabbaa	q (q-1)
$\mathbb{C}_5$	abbaaa	q (q-1)
$\mathbb{C}_6$	bbaaaa	q (q-1)
$\mathbb{C}_7$	abbbba	q (q-1)
$\mathbb{C}_8$	aabbcc	q (q-1) (q-2)
$\mathbb{C}_9$	abbcca	q (q-1) (q-2)
$\mathbb{C}_{10}$	abbacc	q (q-1) (q-1)
$\mathbb{C}_{11}$	bbacca	q (q-1) (q-1)
$\mathbb{C}_{12}$	abccba	q (q-1) (q-1)

Table 1: Equivalence classes under recoloring symmetry of the qDDE model in the null sector for $L = 6$. Here $a, b, c, \dots$ represent different colors. Colors which are adjacent but represented by different symbols are assumed to be distinct. For example in $\mathbb{C}_{10}$, $a \neq b$ and $a \neq c$, but b and c need not be distinct.
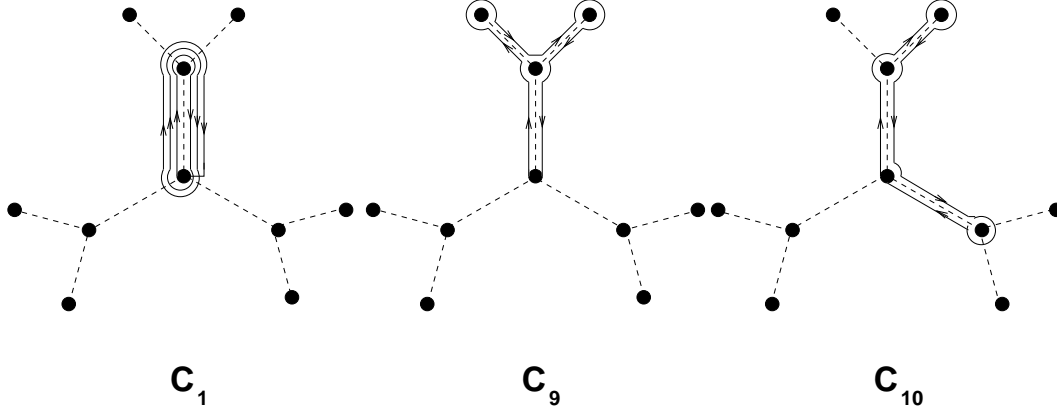
Figure 4: Polymer chain configurations on the Bethe lattice corresponding to equivalence classes $\mathbb{C}_1$, $\mathbb{C}_9$ and $\mathbb{C}_{10}$ of the qDDE model on a chain of length $L = 6$. For every site only 3 bonds are shown.

classes. In general if the polymer chain occupy n distinct bonds of the Bethe lattice, then the number of configurations in the corresponding equivalence class is approximately q^n for large q. As in the steady state all configurations appear with equal probability, the weight of an equivalence class in the steady state is directly proportional to the number of configurations in it. This implies that in the $q \to \infty$ limit, only those equivalence classes where the polymer chain traverse each bond of the Bethe lattice exactly twice or not at all, will have a nonzero weight in the steady state. For example in the $L = 6$ case, only the equivalence classes $\mathbb{C}_8 - \mathbb{C}_{12}$ will have a nonzero weight in the steady state in the $q \to \infty$ limit. If the state of the chain is examined at some instant in the steady state, with probability 1 it will belong to one of the equivalence classes $\mathbb{C}_8 - \mathbb{C}_{12}$. And the weight of all these equivalence classes are the same in this limit, which we can take as 1. Let $\mathbb{L}$ denotes any equivalence class that has weight 1 in the steady state and $\mathbb{S}$ denotes any equivalence class whose weight tends to 0 in the steady state, in the limit of large q.

In the steady state, a finite fraction of spins can flip at any instant. For the spin at site i to flip it is necessary that at least one of its neighbors (at site $i - 1$ or $i + 1$) should have the

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same color. The probability for this is given by $P = 2qN_{L-2}/N_L$, where N_L is the number of configurations in the null sector on a lattice of length L. As $N_L \sim [4(q-1)]^{L/2}$ for large L, $P = 1/2$ in the limit $q \to \infty$.

We have already seen that, in the large q limit, with probability 1 the equivalence class of a configuration in the steady state is an $\mathbb{L}$. The average time the system spends in a particular configuration is very small, of order $1/qL$. This is because in a typical configuration there are order of L flippable pairs, and each such pair can go to any of the approximately q other states with rate 1. However most of these transitions are within the same equivalence class. For example, consider a local configuration of a flippable pair $|\ldots abbc \ldots\rangle$. If the pair bb changes its color to d which is different from both a and c, then the resulting configuration is equivalent to the old by recoloring symmetry. On the otherhand the equivalence classes $\mathbb{S}$ are very short lived. A local configuration of the type $|\ldots aaac \ldots\rangle$ within a time of order $1/q$ will revert back to one of the type $|\ldots abbc \ldots\rangle$ or $|\ldots bbac \ldots\rangle$.

Thus, for large q, most of the dynamics of qDDE model involves transitions within an equivalence class of type $\mathbb{L}$. The only transitions allowed from an $\mathbb{L}$ equivalence class to a different equivalence class is to one of type $\mathbb{S}$. These occur with rate of order 1. Let us say that there is an allowed transition from equivalence class $\mathbb{C}$ to one $\mathbb{C}'$, where $\mathbb{C}$ is of type $\mathbb{L}$ and $\mathbb{C}'$ is of type $\mathbb{S}$. $\mathbb{C}'$ being short lived, in a time of order $1/q$ it makes a transition to a $\mathbb{C}''$ of type $\mathbb{L}$. Let us say it goes to longlived classes $\mathbb{C}''_1, \mathbb{C}''_2, \ldots$ with probabilities $p_1\Gamma, p_2\Gamma, \ldots$. In this way, in the limit of large q, for times $>> 1/q$, we have a coarsegrained description of the stochastic evolution of the qDDE system as transitions between different longlived equivalent classes, and this effective dynamics is *Markovian* with specified rates.

6   Equivalence to the Interface Model

We now show that the effective dynamics of the qDDE model in the large q limit is equivalent to the dynamics of the interface model defined in section 2. Let $\mathsf{S}_i(C)$ denotes the substring corresponding to sites from 1 up to and including i of a configuration C of the qDDE model. And let $h_i(C)$ be the length of the IS corresponding to this substring $\mathsf{S}_i(C)$. It is easy to see that h_i's are non-negative integers and $h_{i+1} - h_i = \pm 1$. The set $\{h_i(C)\}$ will be the same for all configurations C in the same equivalence class $\mathbb{C}$. Hence the set $\{h_i(C)\}$ is a function of

only the equivalence class $\mathbb{C}$ and we may write it as $\{h_i(\mathbb{C})\}$. Moreover the correspondence between equivalence classes $\mathbb{L}$ and $\{h_i\}$ is one to one. Hence every equivalence class $\mathbb{L}$ can be uniquely represented by a set $\{h_i\}$. We identify $h_i's$ with the height variables of the interface model defined in Section 2. Then, the transitions between these equivalence classes in the qDDE model give rise to a Markovian time-evolution of the interface model.

We now proceed to derive the transition rates for this interface dynamics and show that they correspond to special values of λ_1, λ_2 and λ_3.

As h_i represents the length of the IS of the substring up to site i, h_i changes only when the spins at sites i and $i+1$ in the qDDE model are flipped. These can happen only if they have the same color, in which case $h_{i-1} = h_{i+1}$. Again, as $|h_{i+1} - h_i| = 1$, the only allowed transitions are of the type

$$\{\ldots h, h-1, h, \ldots\} \quad \longleftrightarrow \quad \{\ldots h, h+1, h, \ldots\}$$

The rates for this transition will depend on the second neighboring heights h_{i-2} and h_{i+2}. To find the transition rates, let S be the local color configuration at the sites $i-2$ to $i+2$. We write S as $s_1 s_2 s_3 s_4 s_5$, where s_1, s_2, s_3, s_4 and s_5 represents the color at sites $i-2, i-1, i, i+1$ and $i+2$ respectively. Each s_i can be any of the colors $a, b, c, d \ldots$ and their different combinations will correspond to different local height configurations $\{h_i\} = \{\ldots, h_{i-2}, h_{i-1}, h_i, h_{i+1}, h_{i+2}, \ldots\}$. We discuss them one by one.

Case I. $\{h_i\} = \{\ldots, h, h+1, h, h+1, h, \ldots\}$

In this case S is given by *abbcc*. Consider the transition $cc \to bb$. As a result S becomes *abbbb*. As explained earlier, this state is very short lived. There are 3 pairs of b that can flip. It is easy to see that flipping the first or the third bb pair does not change h_i. Flipping the middle bb pair increases h_i to $h_i + 2$. Since all the 3 bb pairs have equal chance of flipping, the effective rate for the middle pair to flip is 1/3. There is another possible way for $h_i \to h_i + 2$. Starting from *abbcc*, first bb flips to cc and then the middle cc pair flips. Thus the net effective rate for $h_i \to h_i + 2$ is 2/3 in this case.

Case II. $\{h_i\} = \{\ldots, h+2, h+1, h, h+1, h, \ldots\}$

In this case S is of the type *abcdd* such that to the left of a there is one of each a, b and c with fully reducible substrings in between them. Consider the transition $dd \to cc$. The new S is *abccc*. The flipping of the first cc pair will increase h_i by 2. Whereas flipping of the second

cc pair will not change h_i. Since both these pairs have equal chance of flipping, the rate for $h_i \rightarrow h_i + 2$ is 1/2 in this case.

Case III. $\{h_i\} = \{\ldots, h, h+1, h, h+1, h+2, \ldots\}$

For these heights S is of the type *abbcd*. The only sequence of transitions which changes h_i is first $bb \rightarrow cc$, followed by the flipping of second *cc* pair. As in the previous case the rate for this transition, which increases h_i by 2, is 1/2 for this case.

Case IV. $\{h_i\} = \{\ldots, h+2, h+1, h, h+1, h+2, \ldots\}$

For this case S is of the type *abcde* with no colors equal. Hence the transition rate out of this local configuration is zero.

The remaining four cases corresponds to reverse transitions $(h_i \rightarrow h_i - 2)$ of the above four cases. For example, if the initial heights are $\{h_i\} = \{\ldots, h-2, h-1, h, h-1, h-2, \ldots\}$, h_i can only decrease by 2 and it corresponds to the reverse transition of case I. Arguing as before, it is easy to determine the rates of these reverse transitions, and they are found to be the same as that of the corresponding $h_i \rightarrow h_i + 2$ transition. Thus the qDDE model in the large q limit corresponds to the interface model of section 2 with $\lambda_1 = 2/3, \lambda_2 = \lambda_3 = 1/2$

7   Monte Carlo and Numerical Diagonalization Studies of the Interface Model

We have studied the interface model by using both Monte Carlo simulations and exact numerical diagonalization of the stochastic matrix. Monte Carlo simulations shows that the average height of the interface shows a scaling form

$$\langle h(t, L) \rangle \sim L^{\alpha} f(t/L^z) \tag{12}$$

where L is the length of the lattice. The scaling function $f(x) \rightarrow x^{\alpha/z}$ as $x \rightarrow 0$ and it become a constant in the limit $x \rightarrow \infty$. In section 3 we have shown that $\alpha = 1/2$.

We have done Monte Carlo simulations for various lattice sizes. These are shown in Figure 5. These can be collapsed into a singe curve using the scaling form (12) with $\alpha = 1/2$ and the dynamical exponent $z = 2.5$. This is shown in figure 6.

We have also determined the dynamical exponent z by numerical diagonalization of the stochastic matrix for finite rings and extrapolating the results to infinite L.
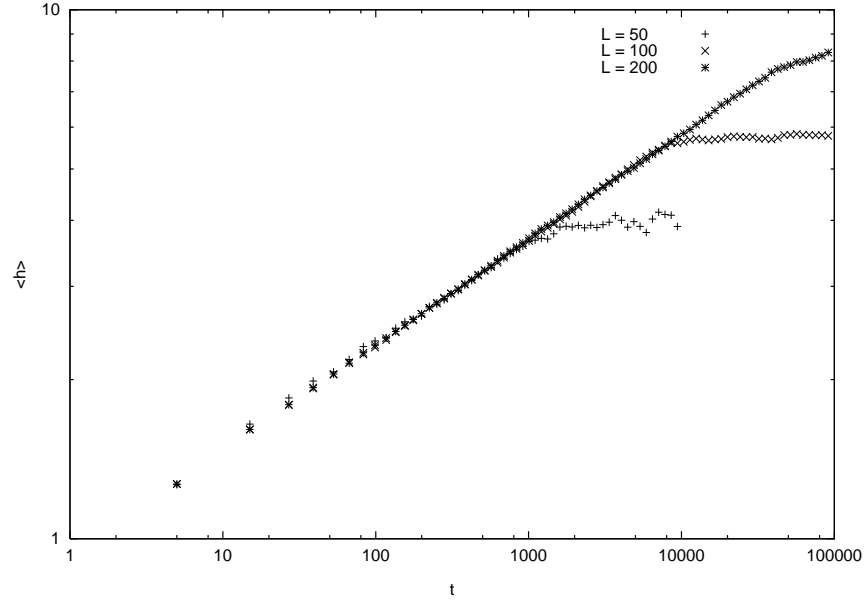
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Figure 5: Average height $\langle h \rangle$ of the interface as a function of time for various lattice sizes.
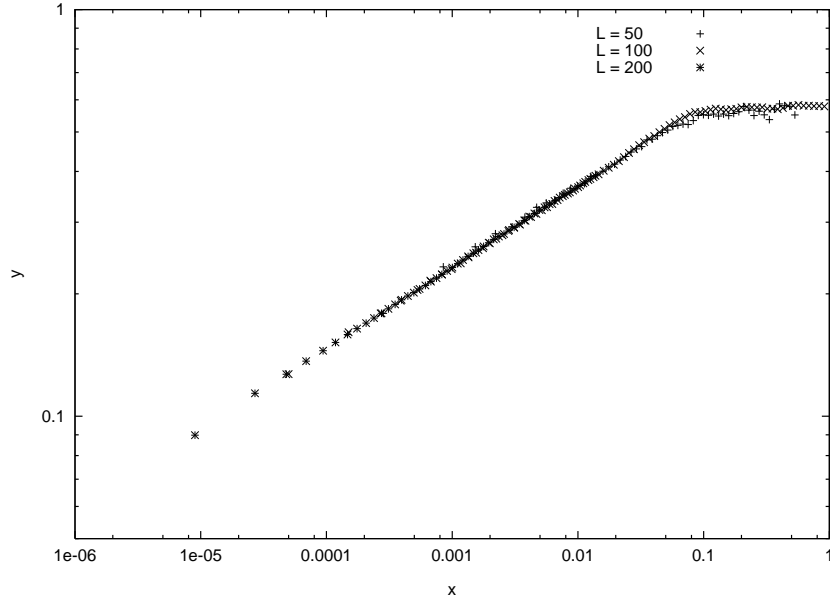

Figure 6: Collapse of the various curves in figure 5. $y = \langle h \rangle L^{-1/2}$ is plotted as a function of the scaling variable $x = t/L^{5/2}$ for various lattice sizes.

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Note that we diagonalized the stochastic matrix of the interface model for a ring of L sites using periodic boundary conditions. This, however, does not correspond to $q \to \infty$ limit of the qDDE model on a ring of size L. This is because the one-to-one correspondence between the equivalence classes of the qDDE model and the interface configurations was proved only with fixed bounadry conditions. In fact, many distinct configurations of the height model on a ring can correspond to same equivalence class under recoloring of the qDDE model on a ring. For example, for $L = 6$, there are only two distinct equivalence classes of qDDE model (corresponding to $\sqsubset$ or Y shaped ring polymers), but the height model with periodic boundary conditions has 10 configurations. The difference between these models should however not be important for large L.

Our procedure of numerical diagonalization is very similar to the one we have used in an earlier study of the TDE model [12]. Here we only briefly out line our procedure. First we reduce the size of the matrix to be diagonalized using symmetries like translation and reflection. Then we find the second largest eigenvalue of the stochastic matrix by numerical diagonalization. Since the largest eigenvalue is zero, this will give the gap Δ in the eigenvalue spectrum. Assuming that the gap scales with system size as $\Delta \sim L^{-z}$, we define an effective dynamical exponent,

$$z_L = \frac{\log[\Delta_{L-2}/\Delta_L]}{\log[L/(L-2)]} \tag{13}$$

where Δ_L is the gap for a ring of size L. The true dynamical exponent z is then obtained by extrapolating z_L to $L = \infty$. We were able to go up to L=20. The size of the matrix to be diagonalized is much smaller than that for the $q = 3$ case of the qDDE model [13], a simplification achieved in the $q \to \infty$ limit. We diagonalized the stochastic matrix for various values of λ_1 keeping $\lambda_2 = \lambda_3 = 1$. The exponent obtained by extrapolating z_L to $L = \infty$ is about 2.8, and it depends on λ_1. In figure (7), we have plotted the values of $\Delta(L)$ versus L on a log-log plot for various values of λ_1. The data donot falls into parallel straight lines. Assuming that this is due to corrections to the scaling of the gap, we tried scaling of the form

$$\Delta(L) = AL^{-z_1} + BL^{-z_2} \tag{14}$$

The best fit were obtained for values of z_1 and z_2 both very close to 2.5. This suggested the extrapolation form

$$\Delta \sim L^{-z}/\log(L/a) \tag{15}$$

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In figure (8) we have plotted $1/(\Delta L^{2.5})$ vs $\log(L)$ for various values of λ_1, keeping $\lambda_2 = \lambda_3 = 1$. Note that the plot involves no fitting parameters, except for the exponent 2.5. The constant a in equation (15) just gives an overall shift to the plot and hence can be taken to be 1. We find that all points falls into straight lines, which gives us some confidence that equation (15) is likely to be the correct scaling form. Note that the $\log(L)$ correction should modify the scaling of the average interface height (equation (12)) to $\langle h(t, L) \rangle \sim L^\alpha f[t/(L^z \log(L/a))]$. We have tried this scaling form for $z = 2.5$ and $a = 1$, and it gave reasonably good collapse, though best collapse seems to be obtained for a slightly lower value of z about 2.4.

To see if the logarithmic correction term is responsible for the poor convergence of observed effective dynamical exponent in the earlier exact diagonalization study of the TDE and 3-color qDDE models [12, 13], we have also plotted earlier data for the 3-color qDDE model in figure (8). The graph is again fairly linear with very small *negative* slope. As a negative slope is not possible asymptotically, we conclude that the 3-DDE model shows no evidence of a logarithmic correction term. Graph for the gap in the TDE model is also quite similar (not shown here). This is consistent with the expectation that these models are in the same universality class, upto logarithmic corrections.

8   Higher Dimensional Generalizations

It is easy to construct higher dimensional interface growth models having the property that minimum of heights is conserved locally. These are however may not realizable as the $q \to \infty$ limit of higher dimensional qDDE models.

Consider an interface model on a square lattice where the heights $h(i, j)$ are integers and nearest neighbor slopes takes values ± 1. The growth rule is that $h(i, j) \to h(i, j) + 2$, if all neighbors have height $h(i, j) + 1$ and at least one of the 4 second neighbors $((i \pm 2, j), (i, j \pm 2))$ has height $h(i, j)$. Reverse transition takes place with the same rate. In this model also $\text{Min}(\{h(i, j)\})$ is conserved. Mountain craters analogue of that in Figure 2 are easy to construct such that it takes a long process of restructuring to go from one to another.

Another simple variant of this model which avoids the odd-even sublattices is the following: We consider an RSOS model with an integer height coordinate $h(i, j)$ at the site (i, j) of a square lattice. The difference in heights at two adjacent sites is constrained to be one of three
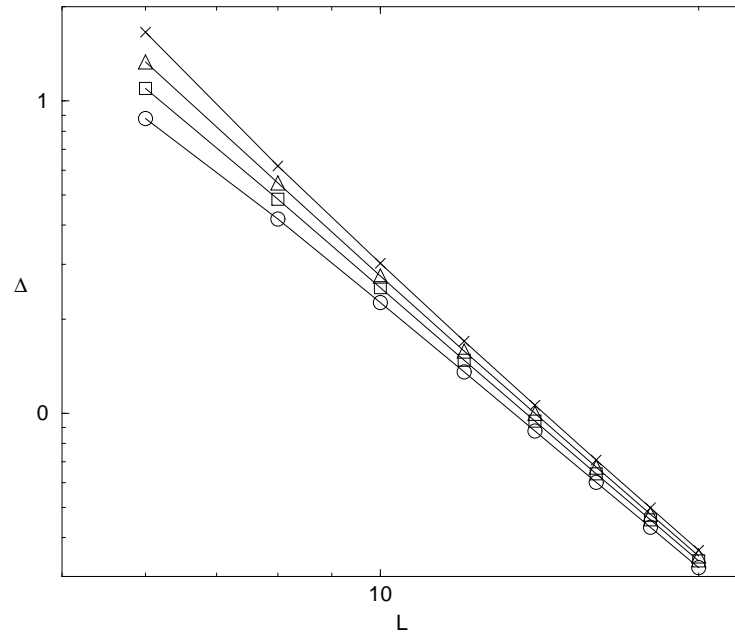
Figure 7: Plot of Δ vs L on a Log-Log scale for $\lambda_2 = \lambda_3 = 1$ and different values of $\lambda_1 = 0.75$ ($\circ$), 1.0 ($\square$), 1.3333 ($\triangle$), 2 ($\times$). Numerically determined values of the slope of these lines are $2.82, 2.90, 2.96$ and 3.05
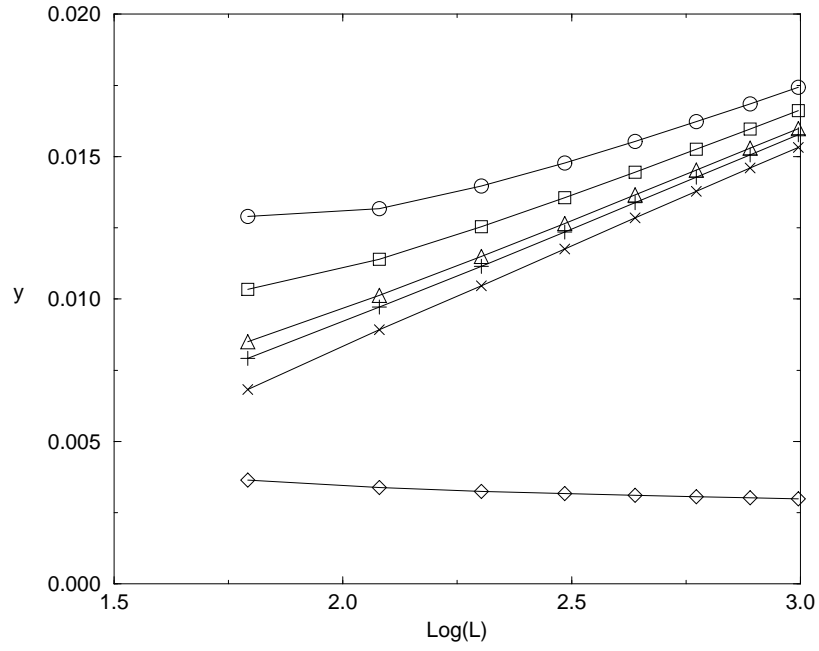
Figure 8: Plot of $y = 1/(\Delta\ L^{2.5})$ vs $\log(L)$ for A) $\lambda_2 = \lambda_3 = 1$ and different values of $\lambda_1 = 0.75$ (○), 1.0 (□), 1.3333 (△), 1.5 (+), 2 (×), B) for the 3-color qDDE model (◇)

values: $-1, 0$ or 1. Thus we allow neighboring sites to have the same height. In the initial configuration, $h(i,j) = 0$ for all sites (i,j). The transition rule is that $h(i,j) \to h(i,j) + 1$ with rate 1, if this would not violate the RSOS constraint, and if at least one of the neighbors have the same height as (i,j). The reverse transition also occurs at the same rate. Clearly this also gives rise to a fluctuating surface whose minimum height value does not change in time. A detailed investigation has not yet been undertaken.

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